Node and Layer Eigenvector Centralities for Multiplex Networks

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Previous attempts
Working on the 3rd order tensor
Experiments

Figure from: http://www.npr.org/2016/04/16/474396452/how-math-determines-the-game-of-thrones-protagonist
Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a complex network with $n$ nodes. Its adjacency matrix is $A \in \mathbb{R}^{n \times n}$:

$$A_{ij} = \begin{cases} 
\omega(i, j) & \text{if } (i, j) \in \mathcal{E} \\
0 & \text{otherwise}
\end{cases}$$
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Figure from: http://complex.unizar.es/jesus/
A multiplex is a collection of graphs

\[
\mathcal{G} = \{ G^{(\ell)} = (V_n, E^{(\ell)}) \}_{\ell=1}^L
\]

all with the same set \( V_n \) of \( n \) nodes, each represented by adj. matrix \( A^{(\ell)} \in \mathbb{R}^{n \times n} \).

The adjacency tensor of the multiplex \( \mathcal{A} \in \mathbb{R}^{n \times n \times L} \) has entries:

\[
A_{ij\ell} = A^{(\ell)}_{ij} = \begin{cases} 
\omega^{(\ell)}_{ij} & \text{if } (i, j) \in E^{(\ell)} \\
0 & \text{otherwise}
\end{cases}
\]
A node centrality measure is a function

\[ C : V_n \rightarrow \mathbb{R}_{\geq 0} \]

which is invariant under graph isomorphism and that assigns to each node a nonnegative score that quantifies its importance within the network.

Bonacich introduced the eigenvector centrality:

\[ x_i = \lambda^{-1} \sum_{j=1}^{n} A_{ij} x_j. \]

It leads to \( Ax = \lambda x \) and, if \( A \) is irreducible, then \( x \) is the Perron vector of \( A \) and \( \lambda = \rho(A) \).
Previous attempts
Matrix-based Complex Networks

Previous attempts

Working on the 3rd order tensor

Experiments
Build the **aggregate matrix**:

\[
A_{\text{agg}}(w) = \sum_{\ell=1}^{L} w_{\ell} A^{(\ell)}. 
\]
Build the aggregate matrix:

\[ A_{\text{agg}} = \sum_{\ell=1}^{L} A^{(\ell)}. \]
Matrix-based

Complex Networks

Build the aggregate matrix:

\[ A_{\text{agg}} = \sum_{\ell=1}^{L} A^{(\ell)}. \]

Then we can define for all \( i \in V_n \):

- **uniform eigenvector-like centrality:**
  \[ \text{agg}_\text{eig}(i) = u_i, \]
  with \( u \) the Perron e-vector of \( A_{\text{agg}}(1) \)

- **aggregate degree:**
  \[ \text{agg}_\text{deg}(i) = e_i^T A_{\text{agg}} \mathbf{1} = \sum_{j=1}^{n} (A_{\text{agg}})_{ij}. \]
Matrix-based

Complex Networks
Previous attempts
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Experiments

SQUEEZE
We work layer by layer:

i) compute the Perron eigenvector $q_\ell$ of layer $\ell$

ii) build the matrix

$$Q = [q_1, q_2, \ldots, q_\ell]$$
Matrix-based Complex Networks

Previous attempts Working on the 3rd order tensor

Experiments

Aggregate the results:

\[ \text{eig}_\text{cen}(i) = (Qw)_i = \sum_{\ell=1}^{L} w_\ell q_\ell(i) \]

We work layer by layer:

1) compute the Perron eigenvector \( q_\ell \) of layer \( \ell \)
2) build the matrix

\[ Q = [q_1, q_2, \ldots, q_\ell] \]
Aggregate the results:

\[ \text{eig}_\text{cen}(i) = (Q1)_i = \sum_{\ell=1}^{L} q_\ell(i). \]

We work layer by layer:

i) compute the Perron eigenvector \( q_\ell \) of layer \( \ell \)

ii) build the matrix

\[ Q = [q_1, q_2, \ldots, q_\ell] \]
Tensor-based Complex Networks
Previous attempts
Working on the 3rd order tensor
Experiments
Use the **supra-adjacency tensor** $\mathcal{B} \in \mathbb{R}^{n \times n \times L \times L}$ and compute

$$
\sum_{j=1}^{n} \sum_{\kappa=1}^{L} \mathcal{B}_{ij\ell\kappa} F_{j\kappa} = \xi F_{i\ell}.
$$

Then the **eigenvector versatility** of node $i$ is defined as

$$
eig\_ver(i) = (F \mathbf{1})_i
$$

To compute it: build the **supra-adjacency matrix**

$$
\mathcal{B} = \\
\begin{bmatrix}
A^{(1)} & I & \cdots & I \\
I & A^{(2)} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \cdots \\
I & \cdots & I & A^{(L)}
\end{bmatrix}
$$

and then compute its Perron eigenvector, which is $\text{vec}(F)$. 
A few downsides of these approaches:

- either too little or “too much” information drawn from the multiplex
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- data needs to be aggregated
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- either too little or “too much” information drawn from the multiplex
- data needs to be aggregated
- uniqueness
Working on the 3rd order tensor
Our approach

The proposed model can be formalized as the solution to the following system of nonlinear equations ($\alpha, \beta > 0$):

\[
\begin{align*}
\sum_{j=1}^{n} \sum_{\ell=1}^{L} A_{ij\ell} x_j t_\ell &= \mu x_i^\alpha \\
\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij\ell} x_i x_j &= \nu t_\ell^\beta
\end{align*}
\]

that has to be fulfilled for some positive scalars $\mu$ and $\nu$.

Recall:
Bonacich's *eigenvector centrality* looks like:

\[
\sum_{j=1}^{n} A_{ij} x_j = \lambda x_i.
\]
Let $\mathcal{A} \in \mathbb{R}_{\geq}^{n \times n \times L}$ and let $\alpha, \beta > 0$ be such that
\[
2\beta^{-1} < (\alpha - 1).
\]

Then, the system has a \textit{unique} solution $(x^*, t^*) \in \mathcal{S}_{\mathcal{A}}^{n \times L}$, where
\[
\mathcal{S}_{\mathcal{A}}^{n \times L} = \left\{ (x, t) \in \mathbb{R}_\geq^n \times \mathbb{R}_\geq^L : \begin{array}{c}
x_i \sim \sum_{j, \ell} A_{ij \ell}, \quad \forall i = 1, \ldots, n \\
t_{\ell} \sim \sum_{i, j} A_{ij \ell}, \quad \forall \ell = 1, \ldots, L \\
||x||_1 = ||t||_1 = 1
\end{array} \right\}
\]
Let:

- $A \in \mathbb{R}_{\geq}^{n \times n \times L}$,
- $\alpha, \beta > 0$ such that $2\beta^{-1} < (\alpha - 1)$ as before, and
- $(x^*, t^*)$ be the unique solution of the system.
- $f = (f_1, f_2) : \mathbb{R}_{\geq}^n \times \mathbb{R}_{\geq}^L \to \mathbb{R}_{\geq}^n \times \mathbb{R}_{\geq}^L$ be defined by

$$f_1(x, t)_i = \left( \sum_{j=1}^{n} \sum_{\ell=1}^{L} A_{ij\ell} x_j t_{\ell} \right)^{1/\alpha},$$

$$f_2(x, t)_{\ell} = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij\ell} x_i x_j \right)^{1/\beta}.$$
For any \((x^{(0)}, t^{(0)}) \in \mathbb{R}^n_+ \times \mathbb{R}^L_+\) consider the iteration

\[
\text{for } k = 0, 1, \ldots \text{ until convergence }\]

\[
\begin{align*}
(x^{(k+1)}, t^{(k+1)}) &= f(x^{(k)}, t^{(k)}), \\
(x^{(k+1)}, t^{(k+1)}) &\leftarrow \frac{(x^{(k+1)}, t^{(k+1)})}{\|(x^{(k+1)}, t^{(k+1)})\|_1}
\end{align*}
\]

Then,

- \(\lim_{k \to \infty} (x^{(k)}, t^{(k)}) = (x^*, t^*)\).

- \(\exists C_1 > 0 \text{ s.t. } \|(x^{(k)}, t^{(k)}) - (x^{(k-1)}, t^{(k-1)})\|_2 \leq C_1 \eta^k, \text{ with } \eta < 1 \text{ known explicitly}\)

- \(\|(x^{(k)}, t^{(k)}) - (x^*, t^*)\|_\infty \leq C_2 \eta^k \text{ with } C_2 \text{ known explicitly.}\)
Let $\mathcal{A} \in \mathbb{R}^{n \times n \times L}_{\geq}$ be a nonzero 3rd-order tensor with undirected layers and let $\alpha, \beta > 0$ be such that $2\beta^{-1} < (\alpha - 1)$. Let $f = (f_1, f_2) : \mathbb{R}^n_\geq \times \mathbb{R}^L_\geq \to \mathbb{R}^n_\geq \times \mathbb{R}^L_\geq$ be defined by

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\]

For any $i \in V_n$ and $\ell \in V_L$, we define
\begin{itemize}
  \item the \textit{f-node eigenvector centrality of node $i$} as $C_f(i) = x_i$, and
  \item the \textit{f-layer eigenvector centrality of layer $\ell$} as $L_f(\ell) = t_\ell$,
\end{itemize}
where $(x, t)$ the unique non-negative eigenvector of $f$ in $S_{\mathcal{A}}^{n \times L}$. 

**Definition**

Let $\mathcal{A} \in \mathbb{R}^{n \times n \times L}_{\geq}$ be a nonzero 3rd-order tensor with undirected layers and let $\alpha, \beta > 0$ be such that $2\beta^{-1} < (\alpha - 1)$. Let $f = (f_1, f_2) : \mathbb{R}^n_\geq \times \mathbb{R}^L_\geq \to \mathbb{R}^n_\geq \times \mathbb{R}^L_\geq$ be defined by

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Experiments

Complex Networks
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EU-air transportation multiplex: $L = 37$ European airlines and $n = 450$ European airports.

- No empty layers.
- All nodes have positive aggregate degree.

**Values of the parameters:** $\alpha = 2.1$ and $\beta = 2$ for all tests, except test 4.

**Stopping criterion:**

$$\max \left\{ \frac{||x^{(k)} - x^{(k-1)}||_2}{||x^{(k)}||_2}, \frac{||t^{(k)} - t^{(k-1)}||_2}{||t^{(k)}||_2} \right\} < 10^{-6}$$

**EUair dataset available from:**

Complex Networks

Previous attempts

Working on the 3rd order tensor

Experiments
### Pearson’s corr. coeff.:

<table>
<thead>
<tr>
<th></th>
<th>$C_f$</th>
<th>eig_ver</th>
<th>eig_cen</th>
<th>agg_eig</th>
<th>agg_deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_f$</td>
<td>-</td>
<td>0.89</td>
<td>0.81</td>
<td>0.86</td>
<td>0.88</td>
</tr>
<tr>
<td>eig_ver</td>
<td>0.89</td>
<td>-</td>
<td>0.97</td>
<td>0.99</td>
<td>0.97</td>
</tr>
<tr>
<td>eig_cen</td>
<td>0.81</td>
<td>0.97</td>
<td>-</td>
<td>0.98</td>
<td>0.94</td>
</tr>
<tr>
<td>agg_eig</td>
<td>0.86</td>
<td>0.99</td>
<td>0.98</td>
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<td>0.97</td>
<td>0.94</td>
<td>0.97</td>
<td>-</td>
</tr>
</tbody>
</table>

### Top 10 ranked nodes:

<table>
<thead>
<tr>
<th></th>
<th>$C_f$</th>
<th>eig_ver</th>
<th>eig_cen</th>
<th>agg_eig</th>
<th>agg_deg</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_f$</td>
<td>50 12 38 40 2 108 252 166 15 57</td>
<td>50 15 40 38 83 2 166 7 64 34</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>eig_ver</td>
<td>40 50 15 83 22 64 14 7 38 2</td>
<td>15 50 83 64 40 38 7 2 166 66</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>agg_eig</td>
<td>15 50 38 40 2 252 64 83 7 12</td>
<td>15 50 38 40 2 252 64 83 7 12</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Definition:** the top $K$ intersection similarity between $x$ and $y$ is defined as

$$\text{isim}_K(x, y) = \frac{1}{K} \sum_{i=1}^{K} \frac{|x_i \Delta y_i|}{2i},$$

where $x_K$ and $y_K$, represent the top $K$ entries of $x$ and $y$, respectively, $\Delta$ is the symmetric difference operator between two sets, and $|S|$ denotes the cardinality of the set $S$. 

![EU air: intersection similarity](image)
Complex Networks

Previous attempts

Working on the 3rd tensor

Experiments

Rankings of nodes

Rankings of layers

Time (sec.)

Number of iterations
Done already:

- Defined a centrality measure for nodes AND layers
- Proved existence and uniqueness under very mild assumptions on the topology of the multiplex
- We have a “power method” iteration to compute it
- The centrality does not rely on aggregate versions of the graph, nor on unfoldings
Wrap up!

Done already:
- Defined a centrality measure for nodes AND layers
- Proved existence and uniqueness under very mild assumptions on the topology of the multiplex
- We have a “power method” iteration to compute it
- The centrality does not rely on aggregate versions of the graph, nor on unfoldings

Future work:
- Include extra (non topological) information on the importance of layers
- Directed layers
- Temporal networks

Paper submitted, soon on arXiv!
Complex Networks

Previous attempts

Working on the 3rd order tensor

Experiments

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Thank you!

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